Int J Theor Phys (2008) 47: 149–167
DOI 10.1007/s10773-007-9547-3

Monoidal Categories, Symmetries, and Compound Physical Systems

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Received: 31 January 2007 / Accepted: 24 August 2007 / Published online: 26 September 2007

Abstract The conceptually non-trivial problem of relating the notion of a compound physical system and the mathematical descriptions of its constituent parts is dramatically illustrated in standard quantum physics by the use of the Hilbert tensor product of the spaces representing the subsystems, instead of the more familiar cartesian product, as it is the case for classical physical systems. Aspects of the general structure of this relationship can be explained by endowing suitable categories that arise in the mathematical descriptions of classical systems and of quantum systems with their natural monoidal structures, and constructing a monoidal functor, relating the monoidal structures of the domain and codomain categories in a coherent way. To highlight some of the structural aspects involved, I will confine myself in this paper to the simple case of finite sets or finite-dimensional Hilbert spaces, on which finite groups act.

1 Introduction

It is a fundamental property of at least some approaches in the Foundations of Physics community, notably the so-called Geneva approach, that the properties of physical systems are described *externally* by way of their relationship with other systems—such as measurement devices used to probe the system under investigation—rather than imposing some putative, a priori structure on the mathematical description ab initio.[1] Philosophically speaking, an important consequence is that a system is typically described by some sort of "top–bottom" approach, since its properties are (at least initially) conceived as originating from the potential interaction of the system as a whole with various experimental contexts. It is conceivable that such an approach may be at odds with the opposite, more reductionistic but conceptually

[1] An important additional advantage, at least from a conceptual point of view, and one that is not often emphasized, is that so-called superselection rules appear ab initio in the framework, and don't need to be introduced ad hoc to save the phenomena.

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straightforward approach of regarding, and trying to describe, a collection of systems as one compound system, from the knowledge of its conceived constituents in isolation. Indeed, the fundamental structural results of Aerts [2] on separated physical systems, in particular quantum systems, makes this incompatibility quite clear. For classical physical systems, both approaches seem equivalent: The information about the parts of the system is sufficient to reconstruct at least the state space associated with the classical system, often as the *cartesian product* $\Sigma_1 \times \Sigma_2$ of the state spaces attributed to the two subsystems, sometimes as an appropriate subset of this, as one may expect from classical logical arguments. For quantum systems, this appears to be no longer the case. It is in this light that we may have to understand the appearance of the *Hilbert tensor product* $\otimes$ in the description of composite quantum systems, rather than a categorical product, in so far as a general quantum system can be described by a complex Hilbert space.

One of the big conceptual advantages of the Geneva Approach (see e.g. [2, 10, 12, 13]), which in a way liberates one from many interpretational issues, comes from the fact that the primitive elements of this formalism are introduced in an operationally well-motivated way. Without going into details, the collection of *properties* attributed to a physical system obtains the structure of a complete atomistic ortholattice. For classical systems, this structure specializes to the collection $\mathcal{P}(\Sigma)$ of all subsets of a set Σ, representing the possible *pure states* attributed to the system, and the standard description of a pure quantum system in a complex Hilbert space $\mathcal{H}$ yields the irreducible orthomodular property lattice of closed subspaces $\mathcal{L}(\mathcal{H})$ of the Hilbert space. Conversely, it is a milestone in research in the Foundations of Physics that similar considerations force every such property lattice to be representable as a collection of ortholattices $\{\mathcal{L}_\omega(\mathcal{H}_\omega) \mid \omega \in \Omega\}$ associated with orthomodular spaces under fairly general conditions [11], where the superselection rules are a manifestation of the existence of classical properties (and this notion is well-defined in the formalism). An orthomodular space comes quite close to a standard Hilbert space, but the nature of the underlying division ring is left undetermined, and additional arguments are necessary to make the transition (see [7], for instance). In addition, we have already indicated that the same structural ingredients, applied to classical physical systems, lead to the structure of the power set of some underlying "state space", the only "structure" of which, at least at this level of the description process, is that of a set. Also here, one needs additional arguments to recover the familiar state spaces from Newtonian or Hamiltonian physics.

The explicit introduction of observables—reflecting the physical desideratum of adding more empiric information in the fundamental framework—together with appropriate symmetry constraints to which these entities may be subject, seems to circumvent the problem associated with viewing a system as compound in some way, since we don't have to take into account the putative a priori existence of proper constituents of the global system. Indeed, one can argue that this is one of the conceptual disadvantages of the Aerts–Daubechies approach [3], where one tries to formulate ab initio a notion of subsystem. Observables are linked to particular experimental devices, and relate the properties of these devices to the properties of the physical system under investigation. Now some physical systems are naturally investigated by a parallel array of such devices, and the mathematical description of the state space associated with the composite, classical device uses the *cartesian* product of the state spaces attributed to the individual set-ups. This appears to be one of the reasons why Abramsky & Coecke [1] are developing a categorical calculus for which the primitive elements are construed as arising from the general experimental possibility of using both serial and parallel operations, subject to appropriate typing constraints, on general physical systems. Thus, the notion of "compoundness" obtains a more *operational* flavor, rather than an invariant attributed ab initio to the system under investigation. A second reason, of course, is that adding more (empiric) structure also leads to important reductions in the possible structure of admissible state spaces, as we shall see.

To highlight some of the structural aspects involved, and for reasons of space, I will confine myself in this paper to the simple case of finite sets or finite-dimensional Hilbert spaces, on which finite groups act,[2] referring more general results associated with the action of locally compact Hausdorff groups on more general quantum-like structures to future publications, in the spirit of [8]. Indeed, the general representation theory has been neglected somewhat in the quantum structures community, although there are notable exceptions (for instance [6]). This has the considerable advantage that we will not be distracted by all kinds of topological and measure-theoretic details, and so we can highlight some of the essential structural aspects, and illustrate how categorical methods can guide our thinking. On the other hand, not all results will remain valid, but this is bound to throw additional light on the general case.

2 Abstract Group Theory

The formalism of category theory is the mathematical language par excellence to investigate structural relationships of all sorts; the reader is referred to the standard literature [4, 9]. Specifically, we claim that many properties of group actions and group representations can be conveniently formulated in a categorical context. Recall that the algebraic identities that make up the definition of a group can be expressed as a collection of commutative diagrams in the category $\underline{\mathrm{Set}}$, that also make sense in other categories, for instance $\underline{\mathrm{Top}}$. In detail, a group object or internal group in a cartesian category $\underline{\mathrm{C}}$ is given by a $\underline{\mathrm{C}}$-object G and $\underline{\mathrm{C}}$-arrows $u : 1 \to G$, $m : G \times G \to G$ and $\zeta : G \to G$, with a specified product for each pair of objects and a specified terminal object 1, such that the conditions expressed in the following diagrams are satisfied.[3]

$$\begin{array}{ccc} G \times (G \times G) & \xrightarrow{G \times m} & G \times G \\ \downarrow{\scriptstyle \alpha} & & \\ (G \times G) \times G & & \downarrow{\scriptstyle m} \\ \downarrow{\scriptstyle m \times G} & & \\ G \times G & \xrightarrow[m]{} & G \end{array} ,$$

[2]For reasons of illumination rather than for support.

[3]Notice that the conditions in the first two diagrams express the notion of an internal monoid, and these conditions also make sense in more general monoidal categories. The third diagram, on the other hand, requires the existence of a *diagonal* $\delta : G \to G \times G$, and this is not generally the case in the more general setting of monoidal categories.

$$\begin{array}{ccccc} 1\times G & \xrightarrow{u\times G} & G\times G & \xleftarrow{G\times u} & G\times 1 \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\rho} & \\ & & G & & \end{array}\quad , \qquad \begin{array}{ccccc} G & \xrightarrow{\delta} & G\times G & \xrightarrow{G\times\zeta} & G\times G \\ {\scriptstyle !}\downarrow & & & & \downarrow{\scriptstyle m} \\ 1 & & \xrightarrow[u]{\qquad\qquad} & & G \end{array}\ ,$$

where α, λ and ρ are the usual (natural) isomorphisms that come with a categorical product. Not surprisingly, there is a corresponding notion of group homomorphism in $\underline{\mathrm{C}}$. Given two internal groups G and G', this is a $\underline{\mathrm{C}}$-arrow $h : G \to G'$ that satisfies the additional requirement

$$\begin{array}{ccc} G\times G & \xrightarrow{m} & G \\ {\scriptstyle h\times h}\downarrow & & \downarrow{\scriptstyle h} \\ G'\times G' & \xrightarrow[m']{} & G' \end{array}\ .$$

In addition, the idea of a group acting on an object can be expressed in a purely diagrammatic way. Explicitly, given a group object G and an object X in $\underline{\mathrm{C}}$, we have the diagrams

$$\begin{array}{ccc} G\times(G\times X) & \xrightarrow{G\times\mathfrak{a}} & G\times X \\ {\scriptstyle\alpha}\downarrow & & \\ (G\times G)\times X & & \downarrow{\scriptstyle\mathfrak{a}} \\ {\scriptstyle m\times X}\downarrow & & \\ G\times X & \xrightarrow[\mathfrak{a}]{} & X \end{array}\ , \qquad \begin{array}{ccc} 1\times X & \xrightarrow{u\times X} & G\times X \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle\mathfrak{a}} \\ & & X \end{array} \tag{1}$$

with $\mathfrak{a} : G\times X \to X$ expressing the action. Finally, the notion of a G-equivariant mapping can also be defined diagrammatically:

$$\begin{array}{ccc} G\times X & \xrightarrow{\mathfrak{a}} & X \\ {\scriptstyle G\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G\times X' & \xrightarrow[\mathfrak{a}']{} & X' \end{array}\ . \tag{2}$$

Consequently, given a group object in a finitely complete category $\underline{C}$, one has a functor

$$P_G : \underline{C} \longrightarrow \underline{C} : \quad \begin{array}{ccc} X & \longmapsto & G \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G\times f} \\ X' & \longmapsto & G \times X' \end{array} . \tag{3}$$

This functor turns out to be part of a monad (P_G, η, μ). Explicitly, the natural transformation $\eta : 1_{\underline{C}} \Rightarrow P_G$ is defined by commutativity of

$$\begin{array}{ccccc} X & \xleftarrow{\lambda} & 1 \times X & \xrightarrow{u\times X} & G \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1\times f} & & \downarrow{\scriptstyle G\times f} \\ X' & \xleftarrow[\lambda]{} & 1 \times X' & \xrightarrow[u\times X']{} & G \times X' \end{array}$$

and the natural transformation $\mu : P_G^2 \Rightarrow P_G$ by commutativity of

$$\begin{array}{ccccc} G \times (G \times X) & \xrightarrow{\alpha} & (G \times G) \times X & \xrightarrow{m\times X} & G \times X \\ {\scriptstyle G\times(G\times f)}\downarrow & & \downarrow{\scriptstyle (G\times G)\times f} & & \downarrow{\scriptstyle G\times f} \\ G \times (G \times X') & \xrightarrow[\alpha]{} & (G \times G) \times X' & \xrightarrow[m\times X']{} & G \times X' \end{array}$$

the monadic conditions reducing to

$$\begin{array}{ccc} G \times (G \times (G \times X)) & \xrightarrow{G\times\mu_X} & G \times (G \times X) \\ {\scriptstyle \mu_{G\times X}}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ G \times (G \times X) & \xrightarrow[\mu_X]{} & G \times X \end{array} ,$$

$$\begin{array}{ccccc} G \times X & \xrightarrow{\eta_{G\times X}} & G \times (G \times X) & \xleftarrow{G\times\eta_X} & G \times X \\ & \searrow^{=} & \downarrow{\scriptstyle \mu_X} & {}^{=}\swarrow & \\ & & G \times X & & \end{array} \tag{4}$$

and the reader can verify that these diagrams commute, because G is an internal group in $\underline{C}$, and the explicit choice of a categorical product and terminal object induce a monoidal structure on $\underline{C}$. We can summarize all this in

Proposition 1 *If* $\underline{\mathrm{C}}$ *is a cartesian category and* G *is an internal* $\underline{\mathrm{C}}$*-group, the triple* (P_G, η, μ) *is a monad on* $\underline{\mathrm{C}}$.

An object in the Eilenberg–Moore category associated with the monadic functor P_G can be seen as an action of G on a $\underline{\mathrm{C}}$-object. More precisely, a P_G-algebra is a pair $(X, \mathfrak{a})$, with $\mathfrak{a} : G \times X \to X$, that satisfies exactly the conditions expressed in (1). A morphism of algebras $(X, \mathfrak{a}) \to (X', \mathfrak{a}')$ is a $\underline{\mathrm{C}}$-arrow that satisfies the condition of G-equivariance (2). This category is then isomorphic to the category $\mathbf{Act}(G, \underline{\mathrm{C}})$, with objects $\underline{\mathrm{C}}$-arrows $G \times X \to X$ that satisfy the previous requirements defining an action, and arrows G-equivariant $\underline{\mathrm{C}}$-arrows.

This procedure works well for monoidal categories for which G can be conceived as an internal group. At first sight, it does not work for monoidal categories such as $\underline{\mathrm{FinHilb}}_{\mathbb{C}}$, consisting of finite-dimensional complex Hilbert spaces and linear operators. More about this later, but first I need some additional stuff.

3 Classical Physical Systems

Before I proceed, let us have a cursory look at some of the mathematical consequences for the description of a classical physical system when a group is known to act on its state space. Here are, in my opinion, two important results, at least at our level of the presentation.

Proposition 2 *If a group* G *acts transitively on a set* X*, then there is a* G*-equivariant bijection*

$$X \cong G/H \tag{5}$$

for some subgroup $H \leq G$*, where* $H = \mathrm{stab}(x_0)$ *for some* $x_0 \in X$.

This representation is essentially independent of x_0.

Proposition 3 *Suppose that* G *acts transitively on* X*. If there exists a* G*-equivariant function* $f : X \to G/H$*, then the structure of the* G*-action* $G \times X \to X$ *is completely determined by the structure of an* H*-action* $H \times S \to S$*, where one can take* $S := f^{-1}(e)$*, and* $H \leq G$*. There exists an equivalence relation* $\sim_H$ *induced by* H *such that*

$$X \cong G \times_H S := G \times S/\sim_H \tag{6}$$

in $\mathbf{Act}(G, \underline{\mathrm{Set}})$*, and* G *acts on the set on the right by left translation*:

$$(g, [a, s]) \mapsto [ga, s]. \tag{7}$$

The first result is well-known, the second may need some clarification. Generally speaking, this result is a manifestation of the existence of a system of imprimitivity relative to $\underline{\mathrm{Set}}$. Specifically, the well-known duality between the categories $\underline{\mathrm{Set}}$ of sets and functions, and $\underline{\mathrm{caBA}}$ of complete atomic Boolean algebras and complete Boolean morphisms, lifts to a duality between the corresponding categories $\mathbf{Act}(G, \underline{\mathrm{Set}})$ and $\mathbf{Act}(G, \underline{\mathrm{caBA}})$. The G-equivariant function $X \to G/H$ corresponds with a G-equivariant complete Boolean morphism

$$\mathcal{P}(G/H) \to \mathcal{P}(X), \tag{8}$$

hence can be interpreted as a G-covariant *observable* for classical orthomodular lattices [12, 13].

Because, at least in the simple case of Set, each action can be seen as an ordinary functor from the group, interpreted as a one-object category, to Set, limits and colimits can be calculated pointwise; in particular, $\mathbf{Act}(G, \underline{\mathrm{Set}})$ has products and coproducts. Specifically, given G-actions $G \times A \to A$ and $G \times B \to B$, we have a G-action on $A \times B$, defined as follows:

$$G \times (A \times B) \to A \times B : (g, (a, b)) \mapsto (g \cdot a, g \cdot b). \tag{9}$$

Similarly, one can verify that G acts on the disjoint union $A \uplus B$, via

$$G \times (A \uplus B) \to A \uplus B : (g, (x, i)) \mapsto (g \cdot x, i), \tag{10}$$

where $i \in \{1, 2\}$. One verifies easily that these prescriptions make the associated projections and coprojections G-equivariant.

4 From Classical Actions to Unitary Representations

So far, we have seen that the category $\mathbf{Act}(G, \underline{\mathrm{FinSet}})$ is finitely complete and cocomplete, and corresponds with the Eilenberg–Moore category associated with the functor P_G. We are interested in the relation between this category and the category $\mathbf{Rep}(G, \underline{\mathrm{FinHilb}_{\mathbb{C}}})$ of unitary representations of G on finite-dimensional complex Hilbert spaces and G-intertwining linear transformations, and in particular in the relation between various natural monoidal structures that exist in the underlying categories FinSet and $\underline{\mathrm{FinHilb}_{\mathbb{C}}}$: products and coproducts in FinSet, and biproduct and tensorial structures in $\underline{\mathrm{FinHilb}_{\mathbb{C}}}$. The physical motivation to do so comes from the fact that composite classical physical systems are usually described by a categorical product, the state of the compound system being completely specified when we know the states of the parts, whereas for quantum systems that are conceived as composite, one employs the Hilbert tensor product. Consequently, I will consider both FinSet and $\underline{\mathrm{FinHilb}_{\mathbb{C}}}$ as (symmetric) monoidal categories, where the monoidal structures are specified by the cartesian product $\times$ and the Hilbert tensor product $\otimes$, respectively. The fact that these two monoidal structures are intimately and coherently related, is made explicit by the existence of a *monoidal functor* between these two categories. In later work, I will try to extend these results to the physically more interesting case of a (second countable) locally compact Hausdorff group or Lie group acting on various sorts of objects.

Given any finite set X, with cardinality $n < +\infty$, one can associate with X the n-dimensional vector space $\mathbb{C}^X$ of complex-valued functions defined on X.[4] In fact, $\mathbb{C}^X$ can be taylored into a complex Hilbert space, by defining an inner product

$$\langle -, - \rangle : \mathbb{C}^X \times \mathbb{C}^X \to \mathbb{C} : (\varphi, \psi) \mapsto \sum_{x \in X} \varphi(x)^* \psi(x), \tag{11}$$

where $(-)^*$ denotes complex conjugation. If G acts on X, there is a natural associated action of G on $\mathbb{C}^X$:

$$U : G \times \mathbb{C}^X \to \mathbb{C}^X : (g, \psi) \mapsto U(g, \psi) := \psi \circ g^{-1}, \tag{12}$$

[4] A parallel exposition seems to be possible for real Hilbert spaces, at least to the amount where we can avoid any results that apply only to $\mathbb{C}$, such as Schur's lemma.

where we have indulged in some abuse of notation, using the fact that g can be seen as a bijection on X. In fact, each linear mapping

$$U_g : \mathbb{C}^X \to \mathbb{C}^X : \psi \mapsto U(g, \psi) \tag{13}$$

is easily seen to be unitary. Specifically, we have

$$\begin{aligned} \langle U_g(\varphi), U_g(\psi) \rangle &= \sum_{x \in X} (U_g(\varphi))(x)^* \, (U_g(\psi))(x) \\ &= \sum_{x \in X} \varphi(g^{-1} \cdot x)^* \, \psi(g^{-1} \cdot x) \\ &= \sum_{y \in X} \varphi(y)^* \psi(y) = \langle \varphi, \psi \rangle. \end{aligned}$$

It makes sense to try to extend this association of objects into a full-blown functorial relationship between the categories $\underline{\mathrm{FinSet}}$ and $\underline{\mathrm{FinHilb}}_{\mathbb{C}}$. Given a function $f : X \to Y$, there is a canonical correspondence between the vector spaces $\mathbb{C}^Y$ and $\mathbb{C}^X$, given by the assignment

$$\mathbb{C}^f := f^* : \mathbb{C}^Y \to \mathbb{C}^X : \psi \mapsto \psi \circ f \tag{14}$$

and $\mathbb{C}^f$ is obviously a linear operator. It is quite clear that this procedure defines a contravariant functor, say Q. I assert that this functor lifts to a functor between the corresponding categories $\mathbf{Act}(G, \underline{\mathrm{FinSet}})$ and $\mathbf{Rep}(G, \underline{\mathrm{FinHilb}}_{\mathbb{C}})$. For if f is G-equivariant, denoting the actions of G on $\mathbb{C}^X$ and $\mathbb{C}^Y$ by U^X and U^Y respectively, we have, given $\psi \in \mathbb{C}^Y$ and $x \in X$:

$$\begin{aligned} (f^*(U_g^Y(\psi)))(x) &= (U_g^Y(\psi))(f(x)) \\ &= \psi(f(g^{-1} \cdot x)) \\ &= (f^*(\psi))(g^{-1} \cdot x) \\ &= (U_g^X(f^*(\psi)))(x) \end{aligned}$$

and we infer that $f^* \circ U_g^Y = U_g^X \circ f^*$ for all $g \in G$. Summarizing, we have constructed a contravariant functor

$$Q_G : \mathbf{Act}(G, \underline{\mathrm{FinSet}}) \longrightarrow \mathbf{Rep}(G, \underline{\mathrm{FinHilb}}_{\mathbb{C}}) : \quad \begin{array}{ccc} X & \longmapsto & \mathbb{C}^X \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle \mathbb{C}^f} \\ Y & \longmapsto & \mathbb{C}^Y \end{array} . \tag{15}$$

The unitary representations in the image of Q are never irreducible in the finite case. Indeed, the one-dimensional subspace spanned by the vector $\sum_{x \in X} \chi_x \in \mathbb{C}^X$, with χ_x the characteristic function of $\{x\}$, is always invariant. Note, however, that this argument breaks down in the infinite-dimensional case.

It is my purpose to extend Q_G into a monoidal functor, relative to the specified product monoidal structure on $\mathbf{Act}(G, \underline{\mathrm{FinSet}})$ and the tensor product on $\mathbf{Rep}(G, \underline{\mathrm{FinHilb}}_{\mathbb{C}})$. First, however, let us have a look at what happens when G acts non-transitively on X; or, in other words: What happens with coproducts? Given unitary representations U^X and U^Y on $\mathbb{C}^X$ and $\mathbb{C}^Y$ respectively, we have to define an appropriate unitary representation on the direct sum. Of course, we take

$$U : G \times (\mathbb{C}^X \oplus \mathbb{C}^Y) \to \mathbb{C}^X \oplus \mathbb{C}^Y : (g, (f_1, f_2)) \mapsto (U_g^X f_1, U_g^Y f_2). \tag{16}$$

Theorem 1 *If X and Y are two finite G-sets, then*

$$\mathbb{C}^{X \coprod Y} \cong \mathbb{C}^X \oplus \mathbb{C}^Y \tag{17}$$

where the isomorphism lives in the category $\mathbf{Rep}(G, \underline{\text{FinHilb}}_{\mathbb{C}})$.

Proof We have to find an intertwining linear bijection between the vector spaces $\mathbb{C}^{X \coprod Y}$ and $\mathbb{C}^X \oplus \mathbb{C}^Y$. I assert that the linear bijection

$$A : \mathbb{C}^{X \coprod Y} \to \mathbb{C}^X \oplus \mathbb{C}^Y : f \mapsto (f \circ copr_X, f \circ copr_Y)$$

will do. For let $g \in G$, then

$$\begin{aligned} U_g(Af) &= U_g(f \circ copr_X, f \circ copr_Y) \\ &= (U_g^X(f \circ copr_X), U_g^Y(f \circ copr_Y)) \\ &= (f \circ copr_X \circ g^{-1}, f \circ copr_Y \circ g^{-1}), \\ A(V_g f) &= ((V_g f) \circ copr_X, (V_g f) \circ copr_Y) \\ &= (f \circ g^{-1} \circ copr_X, f \circ g^{-1} \circ copr_Y) \end{aligned}$$

with slight abuse of notation, hence the assertion follows. □

Next, we have to find out what happens with products in $\mathbf{Act}(G, \underline{\text{FinSet}})$. First, it is clear that the vector spaces $\mathbb{C}^{X \times Y}$ and $\mathbb{C}^X \otimes \mathbb{C}^Y$ are isomorphic in $\underline{\text{FinHilb}}_{\mathbb{C}}$, since they have the same dimension. Moreover, we have a natural isomorphism between the bifunctors

$$(\underline{\text{FinSet}} \times \underline{\text{FinSet}})^{op} \longrightarrow \underline{\text{FinHilb}}_{\mathbb{C}} : \quad \begin{array}{ccc} (X_1, X_2) & \longmapsto & \mathbb{C}^{X_1 \times X_2} \\ \uparrow {\scriptstyle (f_1, f_2)} & & \downarrow {\scriptstyle \mathbb{C}^{f_1 \times f_2}} \\ (Y_1, Y_2) & \longmapsto & \mathbb{C}^{Y_1 \times Y_2} \end{array} \tag{18}$$

and

$$(\underline{\text{FinSet}} \times \underline{\text{FinSet}})^{op} \longrightarrow \underline{\text{FinHilb}}_{\mathbb{C}} : \quad \begin{array}{ccc} (X_1, X_2) & \longmapsto & \mathbb{C}^{X_1} \otimes \mathbb{C}^{X_2} \\ \uparrow {\scriptstyle (f_1, f_2)} & & \downarrow {\scriptstyle \mathbb{C}^{f_1} \otimes \mathbb{C}^{f_2}} \\ (Y_1, Y_2) & \longmapsto & \mathbb{C}^{Y_1} \otimes \mathbb{C}^{Y_2} \end{array} \tag{19}$$

in the sense that the diagram

$$\begin{array}{ccc} \mathbb{C}^{X_1\times Y_1} & \xrightarrow{\varphi_{X_1,Y_1}} & \mathbb{C}^{X_1}\otimes\mathbb{C}^{Y_1} \\ {\scriptstyle \mathbb{C}^{f_1\times f_2}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{C}^{f_1}\otimes\mathbb{C}^{f_2}} \\ \mathbb{C}^{X_2\times Y_2} & \xrightarrow[\varphi_{X_2,Y_2}]{} & \mathbb{C}^{X_2}\otimes\mathbb{C}^{Y_2} \end{array} \tag{20}$$

commutes, where

$$\varphi_{X,Y}:\mathbb{C}^{X\times Y}\to\mathbb{C}^{X}\otimes\mathbb{C}^{Y}:\chi_{(x,y)}\mapsto\chi_x\otimes\chi_y \tag{21}$$

is defined on basis elements, and in general by linear extension of this prescription. Indeed, chasing an arbitrary basis element around the diagram, we obtain $\chi_{(x,y)}\mapsto\chi_x\otimes\chi_y\mapsto\chi_{f_1^{-1}(x)}\otimes\chi_{f_2^{-1}(y)}$ when we go clockwise, and

$$\chi_{(x,y)}\mapsto\chi_{f_1^{-1}(x)\times f_2^{-1}(y)}\mapsto\sum_{a\in f_1^{-1}(x),\ b\in f_2^{-1}(y)}\chi_a\otimes\chi_b=\chi_{f_1^{-1}(x)}\otimes\chi_{f_2^{-1}(y)}$$

in the other direction.

Second, we obviously have an isomorphism $\beta_0:\mathbb{C}\cong\mathbb{C}^1$, where $1:=\{0\}$ is a specified terminal object in FinSet. Let β_2 denote the natural isomorphism φ^{-1}. For the triple (Q,β_2,β_0) to qualify as a strong monoidal functor, one needs to check a bunch of coherence criteria. Specifically, we have to verify commutativity of the following diagrams (where I have omitted the indices of the natural transformations, for reasons of typographical clarity):

$$\begin{array}{ccc} \mathbb{C}^X\otimes(\mathbb{C}^Y\otimes\mathbb{C}^Z) & \xrightarrow{\alpha'} & (\mathbb{C}^X\otimes\mathbb{C}^Y)\otimes\mathbb{C}^Z \\ {\scriptstyle \mathbb{C}^X\otimes\beta_2}\big\downarrow & & \big\downarrow{\scriptstyle \beta_2\otimes\mathbb{C}^Z} \\ \mathbb{C}^X\otimes\mathbb{C}^{Y\times Z} & & \mathbb{C}^{X\times Y}\otimes\mathbb{C}^Z \\ {\scriptstyle \beta_2}\big\downarrow & & \big\downarrow{\scriptstyle \beta_2} \\ \mathbb{C}^{X\times(Y\times Z)} & \xrightarrow[\mathbb{C}^{\alpha^{-1}}]{} & \mathbb{C}^{(X\times Y)\times Z} \end{array}, \tag{22}$$

$$\begin{array}{ccc} \mathbb{C}^X\otimes\mathbb{C} & \xrightarrow{\rho'} & \mathbb{C}^X \\ {\scriptstyle \mathbb{C}^X\otimes\beta_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{C}^{\rho^{-1}}} \\ \mathbb{C}^X\otimes\mathbb{C}^1 & \xrightarrow[\beta_2]{} & \mathbb{C}^{X\times 1} \end{array}, \qquad \begin{array}{ccc} \mathbb{C}\otimes\mathbb{C}^X & \xrightarrow{\lambda'} & \mathbb{C}^X \\ {\scriptstyle \beta_0\otimes\mathbb{C}^X}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{C}^{\lambda^{-1}}} \\ \mathbb{C}^1\otimes\mathbb{C}^X & \xrightarrow[\beta_2]{} & \mathbb{C}^{1\times X} \end{array} \tag{23}$$

with α,α',λ and ρ the obvious natural isomorphisms. It is straightforward to chase basis vectors around the diagrams, and to check that commutativity holds indeed.

Third, we have to verify that the same coherence relations continue to hold in the categories $\mathbf{Act}(G, \underline{\text{FinSet}})$ and $\mathbf{Rep}(G, \underline{\text{FinHilb}}_{\mathbb{C}})$. Therefore, we have to check that all the arrows involved are G-equivariant functions or intertwining linear operators, respectively. First, however, we have to define an appropriate unitary representation of G on $\mathbb{C}^X \otimes \mathbb{C}^Y$ and on $\mathbb{C}$, given unitary representations U^X of G on $\mathbb{C}^X$ and U^Y on $\mathbb{C}^Y$. As for the first, it is natural to take the unitary representation

$$g \mapsto U^X_g \otimes U^Y_g. \tag{24}$$

For the second, we take $1 \mapsto 1$ for all $g \in G$. Continuing our argument,

$$\begin{aligned} g \cdot \alpha(x, (y, z)) &= g \cdot ((x, y), z) \\ &= ((g \cdot x, g \cdot y), g \cdot z) \\ &= \alpha(g \cdot x, (g \cdot y, g \cdot z)) = \alpha(g \cdot (x, (y, z))), \\ g \cdot \lambda(0, x) &= g \cdot x \\ &= \lambda(0, g \cdot x) = \lambda(g \cdot (0, x)) \end{aligned}$$

and similarly for ρ, so these arrows lift to $\mathbf{Act}(G, \underline{\text{FinSet}})$. Next, at the level of $\underline{\text{FinHilb}}_{\mathbb{C}}$ we obtain

$$\begin{aligned} (U_g \circ \alpha')(\chi_x \otimes (\chi_y \otimes \chi_z)) &= ((U^X_g \otimes U^Y_g) \otimes U^Z_g)((\chi_x \otimes \chi_y) \otimes \chi_z) \\ &= (\chi_{g\cdot x} \otimes \chi_{g\cdot y}) \otimes \chi_{g\cdot z} \\ &= \alpha'(\chi_{g\cdot x} \otimes (\chi_{g\cdot y} \otimes \chi_{g\cdot z})) \\ &= (\alpha' \circ (U^X_g \otimes (U^Y_g \otimes U^Z_g))(\chi_x \otimes (\chi_y \otimes \chi_z)), \end{aligned}$$

$$\begin{aligned} (U^X_g \circ \lambda')(1 \otimes \chi_x) &= U^X_g(\chi_x) \\ &= \chi_{g\cdot x} \\ &= \lambda'(1 \otimes \chi_{g\cdot x}) \\ &= \lambda'((U^{\mathbb{C}}_g \otimes U^X_g)(1 \otimes \chi_x)), \\ (V_g \circ \beta_2)(\chi_x \otimes \chi_y) &= V_g(\chi_{(x,y)}) \\ &= \chi_{(g\cdot x, g\cdot y)} \\ &= \beta_2(\chi_{g\cdot x} \otimes \chi_{g\cdot y}) \\ &= \beta_2((U^X_g \otimes U^Y_g)(\chi_x \otimes \chi_y)), \\ (U^1_g \circ \beta_0)(1) &= U^1_g(\chi_0) \\ &= \chi_0 \\ &= \beta_0(1) \\ &= \beta_0(U^{\mathbb{C}}_g(1)) \\ &= (\beta_0 \circ U^{\mathbb{C}}_g)(1) \end{aligned}$$

and this proves our assertion. Summarizing, we have established

Theorem 2 *The strong monoidal functor*

$$(Q, \beta_2, \beta_0) : (\underline{\text{FinSet}}, \times) \longrightarrow (\underline{\text{FinHilb}_{\mathbb{C}}}, \otimes) \tag{25}$$

lifts to a strong monoidal functor

$$(Q_G, \beta_2, \beta_0) : (\mathbf{Act}(G, \underline{\text{FinSet}}), \times) \longrightarrow (\mathbf{Rep}(G, \underline{\text{FinHilb}_{\mathbb{C}}}), \otimes). \tag{26}$$

From a physical perspective, these results point at the importance of the tensor product for the description of compound quantum systems, at least in so far as they can be represented as objects in the category $\underline{\text{FinHilb}_{\mathbb{C}}}$. The parallel composition of two classical measurement devices is described by the cartesian product of the descriptions of the individual devices, and the composite quantum system by the tensor product of the elementary systems that can be probed by the measurement devices individually.

Next, I propose to investigate the Hilbert space equivalent of the characterization of classical actions of a group G, in terms of the Eilenberg–Moore category associated with the functor P_G. Since the functor Q maps $G \times X$ to $\mathbb{C}^{G\times X} \cong \mathbb{C}^G \otimes \mathbb{C}^X$, we expect that tensoring with the *group algebra* $\mathbb{C}^G$ plays an important role in our setting.[5] This algebra comes naturally equipped with a *convolution* product, an operation that reflects the underlying group structure. Specifically, given $\alpha, \beta \in \mathbb{C}^G$ one defines the operation

$$\alpha \star \beta = \left(\sum_{g\in G} \alpha(g)\, \chi_g\right) \star \left(\sum_{h\in G} \beta(h)\, \chi_h\right) = \sum_{g,h\in G} \alpha(g)\beta(h)\chi_{gh} \tag{27}$$

and so

$$(\alpha \star \beta)(k) = \sum_{gh=k} \alpha(g)\beta(h) = \sum_{g\in G} \alpha(kg^{-1})\beta(g) = \sum_{g\in G} \alpha(g)\beta(g^{-1}k). \tag{28}$$

The unit for the convolution product is the element χ_e. Convolution is compatible with the linear structure of $\mathbb{C}^G$, and the reader can easily verify that $\star$ is an associative operation. In addition, the group algebra comes naturally equipped with an involution, defined on generators as $\chi_g \mapsto \chi_{g^{-1}}$, and in general by the assignment

$$\alpha = \sum_{g\in G} \alpha(g)\chi_g \mapsto \alpha^\dagger = \sum_{g\in G} \alpha(g)^* \chi_{g^{-1}}. \tag{29}$$

Incidentally, $(\alpha \star \beta)^\dagger = \beta^\dagger \star \alpha^\dagger$. Indeed, this follows from

$$\begin{aligned}(\alpha \star \beta)^\dagger(k^{-1}) &= \sum_{g\in G} \alpha(g)^* \beta(g^{-1}k)^* \\ &= \sum_{g'\in G} \beta(g'^{-1})^* \alpha(kg')^* = (\beta^\dagger \star \alpha^\dagger)(k^{-1}).\end{aligned}$$

For our purposes it is more than sufficient that the convolution product reflects the underlying group structure, and turns $\mathbb{C}^G$ into an associative $*$-algebra with unit.

[5] Actually, the group algebra is usually taken as the space $\mathrm{L}^1(G, \mu_G)$, which becomes an involutive and associative Banach algebra when equipped with $\star$, but this is irrelevant for our purposes here.

In some sense, the structure of the group algebra $\mathbb{C}^G$ allows us to "internalize" the group G as a group-like object in the category $\underline{\text{FinHilb}}_{\mathbb{C}}$. For, according to the perspective developed in this exposition, it is natural to consider the structure of the functor

$$T_G : \underline{\text{FinHilb}}_{\mathbb{C}} \longrightarrow \underline{\text{FinHilb}}_{\mathbb{C}} : \begin{array}{ccc} V & \longmapsto & \mathbb{C}^G \otimes V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbb{C}^G \otimes f} \\ W & \longmapsto & \mathbb{C}^G \otimes W \end{array} . \tag{30}$$

First, observe that $\mathbb{C}^G$ behaves as a monoid in the category $\underline{\text{FinHilb}}_{\mathbb{C}}$. Specifically, because of the bilinearity of $\star : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}^G$ there exists a unique linear mapping $\hat{\star} : \mathbb{C}^G \otimes \mathbb{C}^G \to \mathbb{C}^G$ such that $\hat{\star} \circ \otimes = \star$, by the universal properties of $\otimes$. We then have to check commutativity of the diagram

$$\begin{array}{ccc} (\mathbb{C}^G \otimes \mathbb{C}^G) \otimes \mathbb{C}^G & \xrightarrow{\hat{\star} \otimes \mathbb{C}^G} & \mathbb{C}^G \otimes \mathbb{C}^G \\ {\scriptstyle \alpha^{-1}}\downarrow & & \\ \mathbb{C}^G \otimes (\mathbb{C}^G \otimes \mathbb{C}^G) & & \downarrow{\scriptstyle \hat{\star}} \\ {\scriptstyle \mathbb{C}^G \otimes \hat{\star}}\downarrow & & \\ \mathbb{C}^G \otimes \mathbb{C}^G & \xrightarrow[\hat{\star}]{} & \mathbb{C}^G \end{array}$$

and it is trivial to verify this on basis elements. It is equally easy to verify the conditions expressed in the two remaining diagrams

$$\begin{array}{ccccc} \mathbb{C} \otimes \mathbb{C}^G & \xrightarrow{u \otimes \mathbb{C}^G} & \mathbb{C}^G \otimes \mathbb{C}^G & \xleftarrow{\mathbb{C}^G \otimes u} & \mathbb{C}^G \otimes \mathbb{C} \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \hat{\star}} & & \downarrow{\scriptstyle \rho} \\ \mathbb{C}^G & = & \mathbb{C}^G & = & \mathbb{C}^G \end{array} ,$$

where u is the linear mapping defined by the assignment $1 \mapsto \chi_e$. We deduce that the functor (30) is part of a monad, explicitly given by the triple (T_G, η, μ), where the natural transformations

$$\eta : 1_{\underline{\text{FinHilb}}_{\mathbb{C}}} \Longrightarrow T_G, \qquad \mu : T_G^2 \Longrightarrow T_G \tag{31}$$

are given by the assignments

$$\eta_V : V \to \mathbb{C}^G \otimes V : \psi \mapsto \chi_e \otimes \psi, \tag{32}$$

$$\mu_V : \mathbb{C}^G \otimes (\mathbb{C}^G \otimes V) \to \mathbb{C}^G \otimes V : \alpha \otimes (\beta \otimes \psi) \mapsto (\alpha \star \beta) \otimes \psi \tag{33}$$

on generators. More explicitly, to qualify as a monad, this triple has to satisfy the conditions

$$\begin{array}{ccc} \mathbb{C}^G \otimes (\mathbb{C}^G \otimes (\mathbb{C}^G \otimes V)) & \xrightarrow{\mathbb{C}^G \otimes \mu_V} & \mathbb{C}^G \otimes (\mathbb{C}^G \otimes V) \\ {\scriptstyle \mu_{\mathbb{C}^G \otimes V}} \downarrow & & \downarrow {\scriptstyle \mu_V} \\ \mathbb{C}^G \otimes (\mathbb{C}^G \otimes V) & \xrightarrow[\mu_V]{} & \mathbb{C}^G \otimes V \end{array} ,$$

$$\begin{array}{ccccc} \mathbb{C}^G \otimes V & \xrightarrow{\eta_{\mathbb{C}^G \otimes V}} & \mathbb{C}^G \otimes (\mathbb{C}^G \otimes V) & \xleftarrow{\mathbb{C}^G \otimes \eta_V} & \mathbb{C}^G \otimes V \\ & \searrow\!\!= & \downarrow {\scriptstyle \mu_V} & =\!\!\swarrow & \\ & & \mathbb{C}^G \otimes V & & \end{array} .$$

The first diagram boils down to associativity of the convolution, and the second to the fact that χ_e acts as an identity with respect to $\star$.

In the next step, we construct the Eilenberg–Moore category $\underline{\mathrm{EM}}(T_G)$ of all T_G-algebras, associated with the monadic functor T_G. It will then hardly come as a surprise that we have the following

Theorem 3 **Rep**$(G, \underline{\mathrm{FinHilb}}_{\mathbb{C}})$ *is a full subcategory of* $\underline{\mathrm{EM}}(T_G)$.

Proof Given a unitary representation $g \mapsto U_g^V$ of G on a finite-dimensional Hilbert space V, we start by showing that the pair $(V, \hat{U}^V)$ forms a T_G-algebra, where $\hat{U}^V$ is defined on generators as follows:

$$\hat{U}^V : \mathbb{C}^G \otimes V \to V : \chi_g \otimes \psi \mapsto U_g^V(\psi)$$

and on arbitrary elements by linear extension. We have to check the conditions expressed in the diagrams

$$\begin{array}{ccc} \mathbb{C}^G \otimes (\mathbb{C}^G \otimes V) & \xrightarrow{\mathbb{C}^G \otimes \hat{U}^V} & \mathbb{C}^G \otimes V \\ {\scriptstyle \mu_V} \downarrow & & \downarrow {\scriptstyle \hat{U}^V} \\ \mathbb{C}^G \otimes V & \xrightarrow[\hat{U}^V]{} & V \end{array} , \qquad \begin{array}{ccc} V & \xrightarrow{\eta_V} & \mathbb{C}^G \otimes V \\ & =\!\!\searrow & \downarrow {\scriptstyle \hat{U}^V} \\ & & V \end{array} . \tag{34}$$

Again, it is an easy matter to do so on basis elements. Second, let $A : V \to W$ be an intertwining map. I assert that A can be seen as a morphism of the algebras $(V, \hat{U}^V)$ and

$(W, \hat{U}^W)$. For we have to verify the condition

$$\begin{array}{ccc} \mathbb{C}^G \otimes V & \xrightarrow{\hat{U}^V} & V \\ {\scriptstyle \mathbb{C}^G \otimes A} \downarrow & & \downarrow {\scriptstyle A} \\ \mathbb{C}^G \otimes W & \xrightarrow[\hat{U}^W]{} & W \end{array} \tag{35}$$

and again this is easy to do on generators. On the other hand, any linear transformation that satisfies (35) is intertwining. Altogether, the embedding

$$\mathbf{Rep}(G, \underline{\text{FinHilb}}_{\mathbb{C}}) \longrightarrow \underline{\text{EM}}(T_G) : \quad \begin{array}{ccc} (V, U^V) & \longmapsto & (V, \hat{U}^V) \\ {\scriptstyle A} \downarrow & & \downarrow {\scriptstyle A} \\ (W, U^W) & \longmapsto & (W, \hat{U}^W) \end{array} \tag{36}$$

satisfies all the necessary requirements. □

Conversely, an arbitrary T_G-algebra (V, U) is a pair consisting of a finite-dimensional Hilbert space V and a (necessarily non-zero) linear mapping $U : \mathbb{C}^G \otimes V \to V$ satisfying the conditions (34). We can then define a collection of linear operators

$$U_g : V \to V : \psi \mapsto U(\chi_g \otimes \psi).$$

Expressing the defining conditions for a T_G-algebra, we find

$$\begin{aligned} U_e(\psi) &= U(\chi_e \otimes \psi) \\ &= (U \circ \eta_V)(\psi) \\ &= \mathrm{id}_V(\psi) = \psi, \\ (U_g \circ U_h)(\psi) &= U_g(U(\chi_h \otimes \psi)) \\ &= U(\chi_g \otimes U(\chi_h \otimes \psi)) \\ &= (U \circ (\mathrm{id}_{\mathbb{C}^G} \otimes U))(\chi_g \otimes (\chi_h \otimes \psi)) \\ &= (U \circ \mu_V)(\chi_g \otimes (\chi_h \otimes \psi)) \\ &= U((\chi_g \star \chi_h) \otimes \psi) \\ &= U(\chi_{gh} \otimes \psi) = U_{gh}(\psi). \end{aligned}$$

We deduce that each U_g is invertible, with inverse $U_{g^{-1}}$. If we require in addition that $\|U\| \leq 1$, then it is easy to see that actually $\|U\| = 1$, and we have, for each $g \in G$ and $\psi \in V$:

$$\|U_g(\psi)\|_2 = \|U(\chi_g \otimes \psi)\| \leq \|\chi_g \otimes \psi\|_2 = \|\psi\|_2.$$

Applying this inequality for $U_{g^{-1}}$, we then also have $\|\psi\|_2 \leq \|U_g(\psi)\|_2$, and so $\|U_g\| = 1$ for all $g \in G$, and we have obtained a *unitary* representation of G. Summarizing, if we restrict our attention to linear contractions, and employ the notation $\underline{\text{FinHilb}}_{\mathbb{C}}^{(1)}$) for the resulting subcategory, then each object in $\underline{\text{EM}}(T_G)$ yields an object in $\mathbf{Rep}(G, \underline{\text{FinHilb}}_{\mathbb{C}})$. This result can be seen as yet another manifestation of the fact that it is the *projective* structure of Hilbert space that is important, the linear structure arising because of the fundamental theorems of projective geometry in the representation of the physically more fundamental underlying projective geometries. Summarizing the exposition so far, we have established most of the

Theorem 4 $\mathbf{Rep}(G, \underline{\text{FinHilb}}_{\mathbb{C}}^{(1)})$ *and* $\underline{\text{EM}}(T_G)$ *are isomorphic categories.*

Proof Given a unitary representation (V, U^V), we have

$$\hat{U}_g^V(\psi) = \hat{U}^V(\chi_g \otimes \psi) = U_g^V(\psi).$$

Conversely, given a T_G-algebra (V, U), we have

$$\hat{U}(\chi_g \otimes \psi) = U_g(\psi) = U(\chi_g \otimes \psi),$$

from which our assertion follows. □

Thus, we have a complete characterization of the collection of unitary representations of G in terms of the Eilenberg–Moore category associated with the monadic functor T_G.

5 Action Extensions and Inducing Representations

Finally, I would like to add a few comments on some of the implications for the operations of extending the action of a subgroup H to the action of a larger group G and the corresponding inducing constructions for unitary representations. In the first case, the heart of the problem is related to the properties of the contravariant functor

$$\text{Sub}_{\underline{\text{Grp}_{\underline{C}}}}(G)^{op} \longrightarrow \underline{\text{CAT}} : \quad \begin{array}{ccc} G & \longmapsto & \mathbf{Act}(G, \underline{C}) \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle R_G^H} \\ H & \longmapsto & \mathbf{Act}(H, \underline{C}) \end{array}, \tag{37}$$

where the domain consists of all subgroups of G that are also subobjects in $\underline{C}$ of the $\underline{C}$-group G, arrows corresponding with the inclusions, and where

$$R_G^H : \mathbf{Act}(G, \underline{C}) \longrightarrow \mathbf{Act}(H, \underline{C}) \tag{38}$$

is the obvious restriction of the action of G to an action of the subgroup H on the same object. As befits a free construction, we are then concerned with the possible existence of a left adjoint

$$F_G^H : \mathbf{Act}(H, \underline{C}) \longrightarrow \mathbf{Act}(G, \underline{C}), \tag{39}$$

so that we have a natural isomorphism

$$\hom_{\mathbf{Act}(G,\underline{C})}(F_G^H(A), B) \cong \hom_{\mathbf{Act}(H,\underline{C})}(A, R_G^H(B)). \tag{40}$$

This will always be the case for $\underline{\mathrm{C}} = \underline{\mathrm{Set}}$, as one can show explicitly by the appropriate constructions. Specifically, for each H-set S one constructs a G-set $G \times_H S$ such that the H-equivariant mapping

$$j : S \to R_G^H(G \times_H S) : s \mapsto [1, s] \tag{41}$$

is universal from S to R_G^H; for more details, see Foulis & Wilce [5]. It follows from a standard theorem of category theory ([9], Theorem IV.1.2) that this prescription on objects, $S \mapsto G \times_H S$, can be extended into a functor F_G^H that is left adjoint to R_G^H.

For unitary representations, one has to investigate the properties of a similar functor, restricting a unitary representation of G to the subgroup H:

$$\mathrm{Sub}_{\underline{\mathrm{Grp}}}(G)^{op} \longrightarrow \underline{\mathrm{CAT}} : \quad \begin{array}{ccc} G & \longmapsto & \mathbf{Rep}(G, \underline{\mathrm{FinHilb}_{\mathbb{C}}}) \\ {\scriptstyle i}\uparrow & & \downarrow {\scriptstyle R_G^H} \\ H & \longmapsto & \mathbf{Rep}(H, \underline{\mathrm{FinHilb}_{\mathbb{C}}}) \end{array} \quad . \tag{42}$$

In this case, it turns out that one has to look for a right adjoint to R_G^H, due to the *contravariance* of the functor Q. Specifically, as a first step it makes sense to investigate the potential universal properties of the object $\mathbb{C}^{G \times_H S}$, equipped with the unitary representation specified by Q_G, or *any other suitable member* of its isomorphism class in $\mathbf{Rep}(G, \underline{\mathrm{FinHilb}_{\mathbb{C}}})$. The G-equivariant canonical quotient $q : G \times S \to G \times_H S$ becomes a G-intertwining linear transformation $q^* : \mathbb{C}^{G \times_H S} \to \mathbb{C}^{G \times H}$, and it is easy to see that the functor Q maps surjections to injections. Therefore, $\mathbb{C}^{G \times_H S}$ is G-isomorphic to a linear subspace of $\mathbb{C}^{G \times S}$, and the latter space is isomorphic to the Hilbert space V^G, where $V = \mathbb{C}^S = Q_H(S)$ comes equipped with a unitary representation of the subgroup H. Indeed, this follows from the well-known fact that

$$\Gamma : \hom_{\underline{\mathrm{Set}}}(G \times S, \mathbb{C}) \cong \hom_{\underline{\mathrm{Set}}}(G, \hom(S, \mathbb{C})) : \zeta \mapsto \Gamma\zeta, \tag{43}$$

where $((\Gamma\zeta)(g))(s) = \zeta(g, s)$. Transporting the unitary representation of G (given by the extension of Q to $\mathbf{Act}(G, \underline{\mathrm{Set}})$) to this new vector space, which becomes a Hilbert space when equipped with the scalar product

$$\langle -, - \rangle : V^G \times V^G \to \mathbb{C} : (\varphi, \psi) \mapsto \sum_{g \in G} \langle \varphi(g), \psi(g) \rangle_V, \tag{44}$$

it turns out that $\mathbb{C}^{G \times_H S}$ can be seen as the linear subspace

$$V^{G:H} := \{\varphi \in V^G \mid \varphi(gh) = V_h^{-1}(\varphi(g)) \text{ for all } g \in G, h \in H\}, \tag{45}$$

equipped with the unitary representation $(g, \psi) \mapsto U_g\psi = \psi \circ g^{-1}$; again, for more details, I refer to Foulis & Wilce [5] and Mackey [8]. Under this form, the construction—the representation induced by V—can be easily generalized to arbitrary unitary representations of H on a space V, and not only the ones in the image of Q, and again it turns out that one obtains a universal arrow

$$J : R_G^H(V^{G:H}) \to V \tag{46}$$

in the category $\mathbf{Rep}(H, \underline{\text{FinHilb}_{\mathbb{C}}})$, for each H-representation V. Here, J is the H-intertwining linear surjection that corresponds with the H-equivariant injection j.

$$\begin{array}{ccc} W & & R^H_G(W) \\ \hat{T} \downarrow & , & R^H_G(\hat{T}) \downarrow \quad \searrow^{T} \\ V^{G:H} & & R^H_G(V^{G:H}) \xrightarrow[J]{} V \end{array} . \tag{47}$$

Thus, the inducing construction proves explicitly that a right adjoint I^H_G for the restriction functor R^H_G exists, by applying the same theorem as before. In addition, the Frobenius reciprocity theorem is an easy consequence of the uniqueness requirement, since the (natural) bijection associated with the adjoint pair $R^H_G \dashv I^H_G$

$$\hom(R^H_G W, V) \cong \hom(W, I^H_G V) : T \mapsto \hat{T} \tag{48}$$

is clearly linear, hence the dimensions of both vector spaces, consisting of the appropriate H-intertwining and G-intertwining operators respectively, are equal. Also, extension and induction in stages is straightforward, due to the (essential) uniqueness and compositional behaviour of adjoint functors. Specifically, if $H_1 \leq H_2 \leq H_3$, then $F^{H_1}_{H_2} \dashv R^{H_1}_{H_2}$ and $F^{H_2}_{H_3} \dashv R^{H_2}_{H_3}$, and one also has $F^{H_2}_{H_3} \circ F^{H_1}_{H_2} \dashv R^{H_1}_{H_2} \circ R^{H_2}_{H_3}$, and so there is a G-equivariant bijection

$$H_3 \times_{H_2} (H_2 \times_{H_1} S) \cong H_3 \times_{H_1} S \tag{49}$$

and similarly for I^H_G. To paraphrase Mac Lane [9]: Who wants more?

6 Outlook

In this paper, I have investigated some of the structural issues that arise when one tries to integrate group theoretical methods, reflecting the existence of symmetry constraints at the level of the description of a physical system. The physically important groups—the Galilei group and the Poincaré group among others—come with additional topological, measure-theoretic and differential structure, and this requires a more sophisticated analysis. Also, the operationally motivated structures that one obtains for the description of physical systems lead, inter alia, to complete atomistic orthomodular lattices, with an underlying projective geometry [11, 12]. Consequently, the mathematical representatives of both classical systems and standard quantum systems become objects in the *same* category of such lattices and appropriate morphisms, and so it becomes imperative to endow the resulting category with an action of G. G-equivariant arrows in the underlying category from the subcategory of classical property lattices obtain the physical meaning of G-covariant *observables*, relating the properties of a classical device with those of the system under investigation. In other words, the external functorial correspondence between classical descriptions and standard quantum systems that was the subject of this paper becomes *internalized* in one and the same category, from this perspective. Much remains to be done, and this is the subject of future work.

Acknowledgements This research was carried out during my employment as a Macquarie University Research Fellow, in collaboration with John Corbett. I also thank Bob Coecke for useful though often beer-induced discussions on the physical meaning of monoidal categories in general and for comments on earlier versions of this project in particular.

References

1. Abramsky, S., Coecke, B.: A categorical semantics of quantum protocols. Log. Comput. Sci. (2004). Also available at quant-ph/0402130
2. Aerts, D.: Description of many separated physical entities without the paradoxes encountered in quantum mechanics. Found. Phys. **12**, 1131–1170 (1982)
3. Aerts, D., Daubechies, I.: Physical justification for using the tensor product to describe two quantum systems as one joint system. Helv. Phys. Acta **51**, 661–675 (1978)
4. Borceux, F.: Handbook of Categorical Algebra. Volume 1: Basic Category Theory; Volume 2: Categories and Structures. Cambridge University Press, Cambridge (1994)
5. Foulis, D.J., Wilce, A.: Free extensions of group actions, induced representations, and the foundations of physics. In: Coecke, B., Moore, D., Wilce, A. (eds.) Current Research in Operational Quantum Logic, pp. 139–165. Kluwer Academic, Dordrecht (2000)
6. Gudder, S.P.: Representations of groups as automorphisms on orthomodular lattices and posets. Can. J. Math. **23**, 659–673 (1971)
7. Gudder, S.P., Piron, C.: Observables and the field in quantum mechanics. J. Math. Phys. **12**, 1583–1588 (1971)
8. Mackey, G.W.: Induced Representations of Groups and Quantum Mechanics. Benjamin, New York (1968)
9. Mac Lane, S.: Categories for the Working Mathematician. Springer, New York (1998)
10. Moore, D.: On state spaces and property lattices. Stud. Hist. Philos. Mod. Phys. **30**, 61–83 (1999)
11. Piron, C.: Axiomatique quantique. Helv. Phys. Acta **37**, 439–468 (1964)
12. Piron, C.: Foundations of Quantum Physics. Benjamin, Reading (1976)
13. Piron, C.: Mécanique Quantique. Bases et Applications. Presses Polytechniques et Universitaires Romandes, Lausanne (1990)